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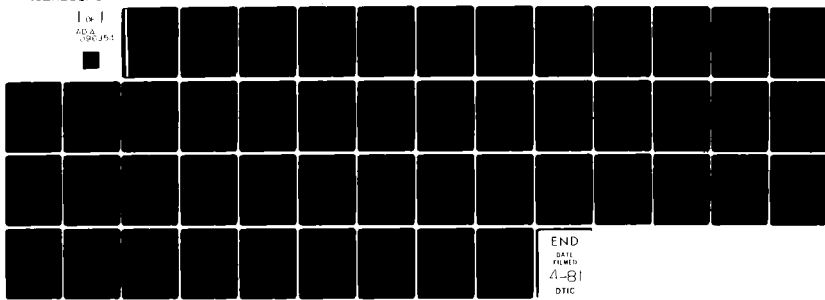
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AN APPLICATION OF IMPLICIT TIME MARCHING TO
THREE DIMENSIONAL FLOW THROUGH A COMPRESSOR BLADE ROW

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Section 1

INTRODUCTION

Three-dimensional flow effects play an important role in the performance of axial-flow fans and compressors that operate at transonic speeds. The coupling between transonic and three-dimensional effects limits the applicability of the two-dimensional analysis methods that have been in use for some years. Efforts to extend these analyses to three-dimensional transonic cases have been aided greatly by the development of computational methods for solving comparable problems in external aerodynamics. The applicable external-flow methods can be divided broadly into two fields: those based on the potential-flow approximation and those that start from the Euler equations. The potential-flow category is further divided into the range of small disturbances and the range where the full nonlinearity of the problem must be accounted for.

The nonlinear small-disturbance potential theory was developed, in a previous AFOSR-sponsored study at Calspan.¹⁻⁵ That work consisted essentially

1. Rae, W.J., "Nonlinear Small-Disturbance Equations for Three-Dimensional Transonic Flow Through a Compressor Blade Row", AFOSR-TR-76-1082, AD-A 51234 (August 1976).
2. Rae, W.J., "Relaxation Solutions for Three-Dimensional Transonic Flow Through a Compressor Blade Row, in the Nonlinear Small-Disturbance Approximation", AFOSR-TR-76-1081, AD-A032553 (August 1976).
3. Rae, W.J., "Finite-Difference Calculations of Three-Dimensional Transonic Flow Through a Compressor Blade Row, Using the Small-Disturbance Nonlinear Potential Equation", pp. 228-252 of Transonic Flow Problems in Turbomachinery, ed. by T.C. Adamson and M.F. Platzer, Hemisphere Publishing Corporation, Washington, (1977).
4. Rae, W.J., "Calculations of Three-Dimensional Transonic Compressor Flow-fields by a Relaxation Method", Journal of Energy, 1 (1977) 284-296.
5. Rae, W.J., "Computer Program for Relaxation Solutions of the Nonlinear Small-Disturbance Equations for Transonic Flow in an Axial Compressor Blade Row", AFOSR-TR-78-0855, AD-A053744 (April 1978).

of an application of the line-relaxation methods and Mach-number-dependent differencing procedures pioneered by Murman and Cole.⁶ Flow field calculations were done, with the resulting computer code, for several blade rows and operating conditions. These calculations showed interesting interactions between the regions of subsonic and supersonic flow that develop within the blade row.

The principal limitation of these results is, of course, the small-disturbance assumption. The pressure ratios and turning angles of practical compressors exceed the range that can properly be called a small perturbation of the inlet conditions. Thus the role of the previous work is chiefly to give qualitative information about the flow.

The present research was undertaken with the aim of extending this earlier work, so as to handle more fully the nonlinearity of the problem. As noted above, the two principal candidates for achieving this goal were the methods for solving the full nonlinear potential equation, and the time-marching methods used for solving the Euler equations. The former approach has the advantage that only a single dependent variable needs to be stored, compared with five dependent variables in the latter case. However, the potential-flow approximation is restricted to isentropic flow. An additional consideration, of considerable importance at the start of this research, was that methods for treating the three-dimensional full potential equation were not yet developed. In contrast, a number of papers describing the "fully implicit" time marching procedure had been published, and appeared to be capable of yielding results in a relatively straightforward way.⁷⁻⁹

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6. Murman, E.M., and Cole, J.D., "Calculation of Plane Steady Transonic Flows", AIAA Journal 9 (1971) 114-121.
 7. Beam, R.M. and Warming, R.F., "An Implicit Finite-Difference Algorithm for Hyperbolic Systems in Conservation-Law Form", J. Comp. Phys. 22 (1976) 87-110.
 8. Steger, J.L., "Implicit Finite Difference Simulation of Flow About Arbitrary Geometries with Application to Airfoils", AIAA Paper 77-665 (June 1977).
 9. Kutler, P., Chakravarthy, S.R., and Lombard, C.P., "Supersonic Flow Over Ablated Nose Tips Using an Unsteady Implicit Numerical Procedure", AIAA Paper 78-213 (January 1978).

Accordingly, the time-marching method was selected for application to the case of flow through an isolated compressor blade row. The adaptations required include a coordinate transformation suitable for a cascade geometry, modifications to enforce mass-flow conservation, boundary conditions upstream and downstream of the blade row, and a means of accounting for the vortex sheets which trail downstream of the blades.

The section below contains a review of the basic equations in absolute and relative coordinates, including two versions of the energy equation. Also described in this section are means for including the radius terms that appear in cylindrical coordinates, and some details about the coordinate transformation used and the metrics that result. The third section is a review of the finite-difference method, patterned after the Beam-Warming technique,⁷ while the fourth section contains a description of the boundary, wake, Kutta, and exit conditions.

All of these elements were incorporated into a computer code, and a number of attempts were made to carry out a sample calculation. These efforts were not successful, due principally to the destabilizing effects of singularities in the metric coefficients at the blade trailing edge, and at the points corresponding to upstream and downstream infinity. Problems arising from the points at infinity were overcome successfully, but no satisfactory resolution of the trailing-edge problem was found.

The section on Concluding Remarks presents some suggestions for further modifications that may be capable of treating the trailing-edge region successfully, and a review of other computer-program elements that will need further development, once the metric-induced instabilities are removed.

Section 2
BASIC EQUATIONS

Conservation-Law Forms

The Euler equations for unsteady three-dimensional flow in cylindrical coordinates x, r, θ may be written in conservation form as (see, for example, Reference 10)

$$\begin{aligned} \frac{\partial}{\partial t}(r\rho) + \frac{\partial}{\partial x}(r\rho V_x) + \frac{\partial}{\partial r}(r\rho V_r) + \frac{\partial}{\partial \theta}(\rho V_\theta) &= 0 \\ \frac{\partial}{\partial t}(r\rho V_x) + \frac{\partial}{\partial x}(r[p + \rho V_x^2]) + \frac{\partial}{\partial r}(r\rho V_x V_r) + \frac{\partial}{\partial \theta}(\rho V_x V_\theta) &= 0 \\ \frac{\partial}{\partial t}(r\rho V_r) + \frac{\partial}{\partial x}(r\rho V_x V_r) + \frac{\partial}{\partial r}(r[p + \rho V_r^2]) + \frac{\partial}{\partial \theta}(\rho V_r V_\theta) &= p + \rho V_\theta^2 \\ \frac{\partial}{\partial t}(r\rho V_\theta) + \frac{\partial}{\partial x}(r\rho V_x V_\theta) + \frac{\partial}{\partial r}(r\rho V_r V_\theta) + \frac{\partial}{\partial \theta}(p + \rho V_\theta^2) &= -\rho V_r V_\theta \\ \frac{\partial}{\partial t}(re) + \frac{\partial}{\partial x}(rV_x[e + p]) + \frac{\partial}{\partial r}(rV_r[e + p]) + \frac{\partial}{\partial \theta}(V_\theta[e + p]) &= 0 \end{aligned} \quad (2-1)$$

where \vec{V} is the absolute velocity, p and ρ are the pressure and density, and e is the total energy per unit volume:

$$e = \rho \left[C_v T + \frac{1}{2} V^2 \right] \quad (2-2)$$

-
10. Vinokur, M., "Conservation Equations of Gasdynamics in Curvilinear Coordinate Systems", Journal of Computational Physics, 14 (1974) pp. 105-125.

These equations are written in the absolute coordinates.¹¹ They may be cast in terms of blade-fixed coordinates by the transformation

$$\begin{aligned} t' &= t, \quad r' = r, \quad \theta' = \theta - \omega t, \quad x' = x \\ w_x &= V_x, \quad w_r = V_r, \quad w_\theta = V_\theta - \omega r \end{aligned} \quad (2-3)$$

where \vec{W} is the velocity relative to the blades. After dropping the primes, these equations have the form:

$$\begin{aligned} \frac{\partial(\rho r)}{\partial t} + \frac{\partial}{\partial x}(r\rho w_x) + \frac{\partial}{\partial r}(r\rho w_r) + \frac{\partial}{\partial \theta}(\rho w_\theta) &= 0 \\ \frac{\partial}{\partial t}(r\rho w_x) + \frac{\partial}{\partial x}(r[p + \rho w_x^2]) + \frac{\partial}{\partial r}(r\rho w_x w_r) + \frac{\partial}{\partial \theta}(\rho w_x w_\theta) &= 0 \\ \frac{\partial}{\partial t}(r\rho w_r) + \frac{\partial}{\partial x}(r\rho w_x w_r) + \frac{\partial}{\partial r}(r[p + \rho w_r^2]) + \frac{\partial}{\partial \theta}(\rho w_r w_\theta) &= p + \rho(w_\theta + \omega r)^2 \\ \frac{\partial}{\partial t}(r\rho w_\theta) + \frac{\partial}{\partial x}(r\rho w_x w_\theta) + \frac{\partial}{\partial r}(r\rho w_r w_\theta) + \frac{\partial}{\partial \theta}(p + \rho w_\theta^2) &= -2\omega r\rho w_r - \rho w_r w_\theta \\ \frac{\partial}{\partial t}(r\rho I) + \frac{\partial}{\partial x}(r w_x \rho I) + \frac{\partial}{\partial r}(r w_r \rho I) + \frac{\partial}{\partial \theta}(w_\theta \rho I) &= \frac{\partial}{\partial t}(r p) \end{aligned} \quad (2-4)$$

where I is the rothalpy:

$$I = c_p T + \frac{w^2}{2} - \frac{1}{2}(\omega r)^2 \quad (2-5)$$

There are two features of these equations that make it difficult to apply the time-marching algorithms developed for external flow. The first is the

11. Wu, C.H., "A General Theory of Three-Dimensional Flow in Subsonic and Supersonic Turbomachines of Axial -, Radial -, and Mixed-Flow Types", NACA TN 2604 (January 1952).

appearance of the time derivative on the right side of the energy equation; in seeking a steady-state solution, it is not clear whether this term should be set equal to zero or evaluated from previous time steps, and the external-flow literature offers no guidance on this question. A preferable form of the energy equation is

$$\frac{\partial}{\partial t}(rK) + \frac{\partial}{\partial x}(rW_x[K+p]) + \frac{\partial}{\partial r}(rW_r[K+p]) + \frac{\partial}{\partial \theta}(W_\theta[K+p]) = 0 \quad (2-6)$$

$$\text{where } K \equiv \rho I - p = \frac{p}{\gamma-1} + \frac{\rho}{2} [W^2 - (\omega r)^2] \quad (2-7)$$

A second awkward feature of these equations is the appearance of the variable r , inserted in several places in order to preserve strict conservation-law form. These appearances require frequent numerical evaluations, most of which can be avoided if the strict conservation form is relaxed slightly, by associating the r -factors only with the θ -derivatives:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho W_x) + \frac{\partial}{\partial r}(\rho W_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho W_\theta) &= - \frac{\rho W_r}{r} \\ \frac{\partial}{\partial t}(\rho W_x) + \frac{\partial}{\partial x}(p + \rho W_x^2) + \frac{\partial}{\partial r}(\rho W_x W_r) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho W_x W_\theta) &= - \frac{\rho W_x W_r}{r} \\ \frac{\partial}{\partial t}(\rho W_r) + \frac{\partial}{\partial x}(\rho W_x W_r) + \frac{\partial}{\partial r}(p + \rho W_r^2) + \frac{1}{r} \frac{\partial}{\partial \theta}(\rho W_r W_\theta) \\ &= - \frac{p + \rho W_r^2}{r} + \frac{p + \rho [W_\theta + \omega r]^2}{r} \\ \frac{\partial}{\partial t}(\rho W_\theta) + \frac{\partial}{\partial x}(\rho W_x W_\theta) + \frac{\partial}{\partial r}(\rho W_r W_\theta) + \frac{1}{r} \frac{\partial}{\partial \theta}(p + \rho W_\theta^2) \\ &= - \frac{\rho W_r W_\theta}{r} - \frac{\rho W_r (2\omega r + W_\theta)}{r} \\ \frac{\partial K}{\partial t} + \frac{\partial}{\partial x}(W_x(K+p)) + \frac{\partial}{\partial r}(W_r(K+p)) + \frac{1}{r} \frac{\partial}{\partial \theta}(W_\theta(K+p)) &= - \frac{W_r}{r}(K+p) \end{aligned} \quad (2-8)$$

Dimensionless Forms

The Euler equations can be made dimensionless in terms of reference values of length, L_{ref} velocity, U_{ref} and density ρ_{ref} , and by dividing the continuity, momentum, and energy equations by $\rho_{ref} U_{ref}$, $\rho_{ref} U_{ref}^2$ and $\rho_{ref} U_{ref}^3$, respectively. Thus each term in the above equations may be considered as a dimensional quantity, or its dimensionless equivalent. The time, for example may be taken in physical units, or in units of L_{ref}/U_{ref} .

These equations can be written in terms of the following five-component vectors:

$$\frac{\partial U}{\partial t} + \frac{\partial E}{\partial x} + \frac{\partial F}{\partial r} + \frac{1}{r} \frac{\partial G}{\partial \theta} = \frac{H - F}{r} \quad (2-9)$$

where

$$U = \begin{pmatrix} \rho \\ \rho W_x \\ \rho W_r \\ \rho W_\theta \\ K \end{pmatrix}, \quad E = \begin{pmatrix} \rho W_x \\ p + \rho W_x^2 \\ \rho W_x W_r \\ \rho W_x W_\theta \\ W_x(K+p) \end{pmatrix}, \quad F = \begin{pmatrix} \rho W_r \\ \rho W_x W_r \\ p + \rho W_r^2 \\ \rho W_r W_\theta \\ W_r(K+p) \end{pmatrix}, \quad (2-10)$$

$$G = \begin{pmatrix} \rho W_\theta \\ \rho W_x W_\theta \\ \rho W_r W_\theta \\ p + \rho W_\theta^2 \\ W_\theta(K+p) \end{pmatrix}, \quad H = \begin{pmatrix} 0 \\ 0 \\ p + \rho(W_\theta + \omega r)^2 \\ -\rho W_r(2\omega r + W_\theta) \\ 0 \end{pmatrix} \quad (2-11)$$

If these equations are subjected to the general coordinate transformation

$$\tau = t, \quad \xi = \xi(x, r, \theta, t), \quad \eta = \eta(x, r, \theta, t), \quad \zeta = \zeta(x, r, \theta, t) \quad (2-12)$$

then, by following Viviani's¹² derivation, it can be shown that the Euler equations retain conservation form, i.e.,

$$\begin{aligned} & \frac{\partial}{\partial \tau} \left(\frac{U}{\mathcal{D}} \right) + \frac{\partial}{\partial \xi} \left(\frac{r[U\xi_t + E\xi_x + F\xi_r] + G\xi_\theta}{r\mathcal{D}} \right) + \frac{\partial}{\partial \eta} \left(\frac{r[U\eta_t + E\eta_x + F\eta_r] + G\eta_\theta}{r\mathcal{D}} \right) \\ & + \frac{\partial}{\partial \zeta} \left(\frac{r[U\zeta_t + E\zeta_x + F\zeta_r] + G\zeta_\theta}{r\mathcal{D}} \right) \\ & = \frac{H-F}{r\mathcal{D}} + \frac{G}{\mathcal{D}} \left\{ \xi_\theta \frac{\partial}{\partial \xi} \left(\frac{1}{r} \right) + \eta_\theta \frac{\partial}{\partial \eta} \left(\frac{1}{r} \right) + \zeta_\theta \frac{\partial}{\partial \zeta} \left(\frac{1}{r} \right) \right\} \\ & = \frac{\partial}{\partial \theta} \left(\frac{1}{r} \right) = 0 \end{aligned} \quad (2-15)$$

where the zero term on the right side results from adding appropriate terms to achieve the conservation form, and where \mathcal{D} is the Jacobian:

$$\mathcal{D} = \frac{\partial(\xi, \eta, \zeta)}{\partial(x, r, \theta)} = \begin{vmatrix} \xi_x & \xi_r & \xi_\theta \\ \eta_x & \eta_r & \eta_\theta \\ \zeta_x & \zeta_r & \zeta_\theta \end{vmatrix} \quad (2-14)$$

These can also be written as

12. Viviani, H., "Formes Conservatives des Equations de la Dynamique des Gaz", La Recherche Aeronautique (1974), No. 1, January-February, pp. 65-66.

$$\frac{\partial \hat{U}}{\partial \tau} + \frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} = \hat{H} \quad (2-15)$$

where

$$\begin{aligned} \hat{U} &= \frac{U}{\mathcal{D}}, \quad \hat{E} = \left[U\xi_t + E\xi_x + F\xi_r + \frac{G\xi_\theta}{r} \right] / \mathcal{D} \\ \hat{F} &= \left[U\eta_t + E\eta_x + F\eta_r + \frac{G\eta_\theta}{r} \right] / \mathcal{D} \\ \hat{G} &= \left[U\zeta_t + E\zeta_x + F\zeta_r + \frac{G\zeta_\theta}{r} \right] / \mathcal{D} \\ \hat{H} &= \frac{H - F}{r\mathcal{D}} \end{aligned} \quad (2-16)$$

Finally, by inserting the definitions of E, F, and G, the vectors \hat{E} , \hat{F} , and \hat{G} can be written as

$$\hat{E} = \frac{1}{\mathcal{D}} \begin{pmatrix} \rho W_1 \\ \rho W_1 W_x + p\xi_x \\ \rho W_1 W_r + p\xi_r \\ \rho W_1 W_\theta + p\xi_\theta/r \\ (K+p)W_1 - p\xi_t \end{pmatrix}, \quad \hat{F} = \frac{1}{\mathcal{D}} \begin{pmatrix} \rho W_2 \\ \rho W_2 W_x + p\eta_x \\ \rho W_2 W_r + p\eta_r \\ \rho W_2 W_\theta + p\eta_\theta/r \\ (K+p)W_2 - p\eta_t \end{pmatrix}, \quad \hat{G} = \frac{1}{\mathcal{D}} \begin{pmatrix} \rho W_3 \\ \rho W_3 W_x + p\zeta_x \\ \rho W_3 W_r + p\zeta_r \\ \rho W_3 W_\theta + p\zeta_\theta/r \\ (K+p)W_3 - p\zeta_t \end{pmatrix} \quad (2-17)$$

where W_1 , W_2 and W_3 are the contravariant components of the velocity vector:

$$W_1 = \xi_t + W_x \xi_x + W_r \xi_r + W_\theta \frac{\xi_\theta}{r}$$

$$W_2 = \eta_t + W_x \eta_x + W_r \eta_r + W_\theta \frac{\eta_\theta}{r}$$

$$W_3 = \zeta_t + W_x \zeta_x + W_r \zeta_r + W_\theta \frac{\zeta_\theta}{r} \quad (2-18)$$

Note that in most places the factor r appears as a divisor of the metrics of the third column of \mathcal{D} , for example as ξ_θ/r ; thus only the ratio ξ_θ/r needs to be stored at each grid point, rather than both factors.

Section 3
FINITE-DIFFERENCE METHOD

The algorithm presented by Beam and Warming⁷ is applied as follows:
(here, the superscript n denotes the time level of the solution):

$$\begin{aligned}\hat{U}^{n+1} &= \hat{U}^n + \frac{\Delta t}{2} \left[\left(\frac{\partial \hat{U}}{\partial t} \right)^n + \left(\frac{\partial \hat{U}}{\partial t} \right)^{n+1} \right] + O(\Delta t)^3 \\ &= \hat{U}^n - \frac{\Delta t}{2} \left[\left(\frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} \right)^n + \left(\frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} \right)^{n+1} \right] \\ &\quad + \frac{\Delta t}{2} \left[\hat{H}^n + \hat{H}^{n+1} \right] + O(\Delta t)^3\end{aligned}\tag{3-1}$$

Next, a Taylor-series expansion is made, i.e.:

$$\begin{aligned}\hat{E}^{n+1} &= \hat{E}^n + A^n (\hat{U}^{n+1} - \hat{U}^n) + O(\Delta t)^2 \\ \hat{F}^{n+1} &= \hat{F}^n + B^n (\hat{U}^{n+1} - \hat{U}^n) + O(\Delta t)^2 \\ \hat{G}^{n+1} &= \hat{G}^n + C^n (\hat{U}^{n+1} - \hat{U}^n) + O(\Delta t)^2 \\ \hat{H}^{n+1} &= \hat{H}^n + D^n (\hat{U}^{n+1} - \hat{U}^n) + O(\Delta t)^2\end{aligned}\tag{3-2}$$

where the coefficients in the expansion are the matrices:

$$\begin{aligned}\hat{A} &= \frac{\partial \hat{E}}{\partial \hat{U}}, \quad \hat{B} = \frac{\partial \hat{F}}{\partial \hat{U}}, \quad \hat{C} = \frac{\partial \hat{G}}{\partial \hat{U}}, \quad \hat{D} = \frac{\partial \hat{H}}{\partial \hat{U}} \\ \hat{E} &= \hat{A} \hat{U}, \quad \hat{F} = \hat{B} \hat{U}, \quad \hat{G} = \hat{C} \hat{U}, \quad \hat{H} = \hat{D} \hat{U}\end{aligned}$$

This enables the equation to be written as

$$\begin{aligned}
\hat{U}^{n+1} - \hat{U}^n &= -\frac{\Delta t}{2} \left\{ 2 \left(\frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} \right)^n + \frac{\partial}{\partial \xi} A^n (\hat{U}^{n+1} - \hat{U}^n) \right. \\
&\quad + \frac{\partial}{\partial \eta} B^n (\hat{U}^{n+1} - \hat{U}^n) + \frac{\partial}{\partial \zeta} C^n (\hat{U}^{n+1} - \hat{U}^n) \left. \right\} \\
&\quad + \left\{ 2H^n + D^n (\hat{U}^{n+1} + \hat{U}^n) \right\} \frac{\Delta t}{2} + O(\Delta t)^3
\end{aligned} \tag{3-4}$$

which can be rearranged as

$$\begin{aligned}
&\left\{ I + \frac{\Delta t}{2} \left(\frac{\partial}{\partial \xi} A^n + \frac{\partial}{\partial \eta} B^n + \frac{\partial}{\partial \zeta} C^n - D^n \right) \right\} \hat{U}^{n+1} \\
&= \left\{ I + \frac{\Delta t}{2} \left(\frac{\partial}{\partial \xi} A^n + \frac{\partial}{\partial \eta} B^n + \frac{\partial}{\partial \zeta} C^n - D^n \right) \right\} \hat{U}^n \\
&\quad - \Delta t \left(\frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} - \hat{H} \right)^n + O(\Delta t)^3
\end{aligned} \tag{3-5}$$

where I is the identity matrix. Certain terms are now added, of order $(\Delta t)^2$ and $(\Delta t)^3$, which have the effect of "completing the cube" on the left-hand side, so that it can be factored as:

$$\begin{aligned}
&\left\{ \left(I + \frac{\Delta t}{2} \frac{\partial}{\partial \xi} A^n \right) \left(I + \frac{\Delta t}{2} \frac{\partial}{\partial \eta} B^n \right) \left(I + \frac{\Delta t}{2} \left[\frac{\partial}{\partial \zeta} C^n - D^n \right] \right) \right\} \hat{U}^{n+1} \\
&= \left\{ \left(I + \frac{\Delta t}{2} \frac{\partial}{\partial \xi} A^n \right) \left(I + \frac{\Delta t}{2} \frac{\partial}{\partial \eta} B^n \right) \left(I + \frac{\Delta t}{2} \left[\frac{\partial}{\partial \zeta} C^n - D^n \right] \right) \right\} \hat{U}^n \\
&\quad - \Delta t \left\{ \frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} - \hat{H} \right\} + O(\Delta t)^3
\end{aligned} \tag{3-6}$$

Following Beam and Warming, this equation is rearranged as

$$\begin{aligned}
& \left(I + \frac{\Delta\tau}{2} \frac{\partial}{\partial \xi} \hat{A}^n \right) \left(I + \frac{\Delta\tau}{2} \frac{\partial}{\partial \eta} \hat{B}^n \right) \left(I + \frac{\Delta\tau}{2} \left[\frac{\partial}{\partial \zeta} \hat{C}^n - \hat{D}^n \right] \right) \Delta \hat{U} \\
& = -\Delta\tau \left\{ \frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} - \hat{H} \right\}^n + O(\Delta\tau)^3
\end{aligned} \tag{3-7}$$

where

$$\Delta \hat{U} = \hat{U}^{n+1} - \hat{U}^n$$

The matrices A, B, C, and D (without the (^) symbol) are defined as

$$\begin{aligned}
A &= \frac{\partial E}{\partial U}, \quad B = \frac{\partial F}{\partial U}, \quad C = \frac{\partial G}{\partial U}, \quad D = \frac{\partial H}{\partial U} \\
E &= AU, \quad F = BU, \quad G = CU, \quad H = DU
\end{aligned} \tag{3-8}$$

The relations between the two sets are

$$\begin{aligned}
\hat{A} &= I \xi_t + A \xi_x + B \xi_r + C \frac{\xi_\theta}{r} \\
\hat{B} &= I \eta_t + A \eta_x + B \eta_r + C \frac{\eta_\theta}{r} \\
\hat{C} &= I \zeta_t + A \zeta_x + B \zeta_r + C \frac{\zeta_\theta}{r} \\
\hat{D} &= \frac{D-B}{r}
\end{aligned} \tag{3-9}$$

The matrices A, B, C, and D are given in the Appendix.

Equation (3-7) is usually referred to as the "delta form" of the algorithm. Its numerical solution is found by a sequence of three one-dimensional solutions:

$$\begin{aligned}
 \left(I + \frac{\Delta \tau}{2} \frac{\partial}{\partial \xi} \hat{A}^n \right) \Delta \hat{U}^* &= -\Delta \tau \left\{ \frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} - \hat{H} \right\}^n \\
 \left(I + \frac{\Delta \tau}{2} \left[\frac{\partial}{\partial \eta} \hat{B}^n - \hat{D}^n \right] \right) \Delta \hat{U}^{**} &= \Delta \hat{U}^* \\
 \left(I + \frac{\Delta \tau}{2} \frac{\partial}{\partial \zeta} \hat{C}^n \right) \Delta \hat{U} &= \Delta \hat{U}^{**}
 \end{aligned} \tag{3-10}$$

Here the term \hat{D} has been placed in the second step, in order to facilitate the calculation of two-dimensional cases, which bypass the radial solution step altogether. This term can be placed in any one of the three steps, with no change in the truncation error.

Damping Terms

The numerical algorithm described above, which uses central differences for the spatial derivatives, requires the addition of certain damping terms for stability. These terms are added by rewriting Eq. 3-10 as follows:

$$\begin{aligned}
 &\left\{ I + \frac{\Delta \tau}{2} \left(\frac{\partial}{\partial \xi} \hat{A}^n - \frac{\epsilon_A (\Delta \xi)^2}{\Delta} \frac{\partial^2}{\partial \xi^2} \Delta \right) \right\} \cdot \left\{ I + \frac{\Delta \tau}{2} \left(\frac{\partial}{\partial \eta} \hat{B}^n - \frac{\epsilon_B (\Delta \eta)^2}{\Delta} \frac{\partial^2}{\partial \eta^2} \Delta \right) \right\} \\
 &\cdot \left\{ I + \frac{\Delta \tau}{2} \left(\frac{\partial}{\partial \zeta} \hat{C}^n - \hat{D}^n - \frac{\epsilon_C (\Delta \zeta)^2}{\Delta} \frac{\partial^2}{\partial \zeta^2} \Delta \right) \right\} \Delta \hat{U} \\
 &= -\Delta \tau \left\{ \frac{\partial \hat{E}}{\partial \xi} + \frac{\partial \hat{F}}{\partial \eta} + \frac{\partial \hat{G}}{\partial \zeta} - \hat{H} \right\}^n \\
 &- \frac{\epsilon_B \Delta \tau}{\Delta} \left\{ (\Delta \xi)^4 \frac{\partial^4}{\partial \xi^4} + (\Delta \eta)^4 \frac{\partial^4}{\partial \eta^4} + (\Delta \zeta)^4 \frac{\partial^4}{\partial \zeta^4} \right\} (\Delta \hat{U}^n) \tag{3-11}
 \end{aligned}$$

The fourth derivatives on the right hand side are evaluated explicitly, using central differences of data at the previous time step. At grid points next to the boundaries, the second derivative is used.

The additional terms on the left side are treated implicitly, i.e., they appear as corrections to the coefficients of the block tridiagonal matrix equations.

Step-Size Considerations

For a fully implicit method, the time step would be limited only by considerations of accuracy, and not by stability. As will be noted below, the present application (and all of the external - aerodynamics literature as well) uses boundary conditions that are explicit, and involves data one time step behind. This introduces the problem of stability considerations, and it is usually found that the time step must be on the order of that given by the Courant-Friedrichs-Lewy condition:

$$\Delta t_{max} \leq \frac{\min(\Delta \xi, \Delta \eta, \Delta \zeta)}{\lambda_{max}} \quad (3-12)$$

where λ denotes an eigenvalue of the matrices that appear in the nonconservative form of the Euler equations. For the present case, these differ only slightly from the rectangular-coordinate set discussed by Warming, Beam, and Hyett.¹³ The non-conservative forms of the equations are

$$\frac{\partial u}{\partial t} + A_x \frac{\partial u}{\partial x} + A_r \frac{\partial u}{\partial r} + \frac{A_\theta}{r} \frac{\partial u}{\partial \theta} = R \quad (3-13)$$

where

13. Warming, R.F., Beam, R.M. and Hyett, B.F., "Diagonalization and Simultaneous Symmetrization of the Gas-Dynamic Matrices", *Mathematics of Computation*, 29, (1975) 1037-1045.

$$u = \begin{pmatrix} \rho \\ w_x \\ w_r \\ w_\theta \\ p \end{pmatrix}, \quad A_x = \begin{pmatrix} w_x & \rho & 0 & 0 & 0 \\ 0 & w_x & 0 & 0 & 1/\rho \\ 0 & 0 & w_x & 0 & 0 \\ 0 & 0 & 0 & w_x & 0 \\ 0 & \rho c^2 & 0 & 0 & w_x \end{pmatrix},$$

$$A_r = \begin{pmatrix} w_r & 0 & \rho & 0 & 0 \\ 0 & w_r & 0 & 0 & 0 \\ 0 & 0 & w_r & 0 & 1/\rho \\ 0 & 0 & 0 & w_r & 0 \\ 0 & 0 & \rho c^2 & 0 & w_r \end{pmatrix}, \quad A_\theta = \begin{pmatrix} w_\theta & 0 & 0 & \rho & 0 \\ 0 & w_\theta & 0 & 0 & 0 \\ 0 & 0 & w_\theta & 0 & 0 \\ 0 & 0 & 0 & w_\theta & 1/\rho \\ 0 & 0 & 0 & \rho c^2 & w_\theta \end{pmatrix}$$

$$R = \begin{pmatrix} -\rho w_r / r \\ 0 \\ (w_\theta + \omega r)^2 / r \\ -(2\omega r + w_\theta) / r \\ 0 \end{pmatrix}, \quad c^2 = \frac{\delta p}{\rho} \quad (3-14)$$

The eigenvalues of A_x , A_r , and A_θ are w_x , w_r , w_θ , $\pm c$, and are related to those of the conservation-law form by the similarity transformation

$$A_j = \mu^{-1} \tilde{A}_j \mu \quad (3-15)$$

where A_j is A_x , A_r , or A_θ , \tilde{A}_j is A , B , or C (see Eq. 3-8), and μ is the Jacobian matrix $\partial u / \partial \mu$:

$$\mu = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ w_x & \rho & 0 & 0 & 0 \\ w_r & 0 & \rho & 0 & 0 \\ w_\theta & 0 & 0 & \rho & 0 \\ \frac{w^2 - \omega^2 r^2}{2} & \rho w_x & \rho w_r & \rho w_\theta & 1/8 - 1 \end{pmatrix} \quad (3-16)$$

$$\mu^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -W_x/\rho & 1/\rho & 0 & 0 & 0 \\ -W_r/\rho & 0 & 1/\rho & 0 & 0 \\ -W_\theta/\rho & 0 & 0 & 1/\rho & 0 \\ \frac{\delta-1}{2}(W^2 + \omega^2 r^2) & -(\delta-1)W_x & -(\delta-1)W_r & -(\delta-1)W_\theta & \delta-1 \end{pmatrix}$$

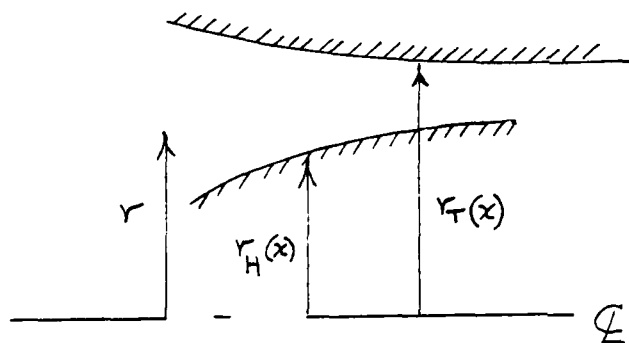
Thus

$$\lambda_{\max} = \max \left\{ k_0 + k_1 W_x + k_2 W_r + k_3 W_\theta, k_0 + k_1 W_x + k_2 W_r + k_3 W_\theta \pm C \sqrt{k_1^2 + k_2^2 + k_3^2} \right\} \quad (3-17)$$

where k_0, k_1, k_2 , and k_3 are either $\xi_t, \xi_x, \xi_r, \frac{\xi_\theta}{r}$, or $\eta_t, \eta_x, \eta_r, \frac{\eta_\theta}{r}$, or $\zeta_t, \zeta_x, \zeta_r, \frac{\zeta_\theta}{r}$.

Coordinate Transformations and Metrics

The geometry of the axisymmetric flow passage is used first, to define the coordinate η as the fractional distance from hub to tip:



$$\eta = \frac{r - r_H(x)}{r_T(x) - r_H(x)} = \eta(x, r)$$

$$\frac{\partial \eta}{\partial \theta} = 0 \quad (3-18)$$

Figure 1.

(The bullet-nose contour of the hub, at some distance upstream of the blade row, must be replaced by some sort of a smooth transition). The intersection of the blade surfaces with the surfaces η - constant defines a two-dimensional cascade:

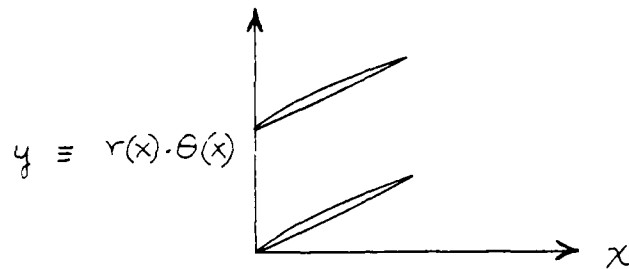


Figure 2.

This cascade is then mapped into a square, using the conformal transformation described in Reference 14. The correspondence of points in the cascade and mapped planes is shown in Fig. 3.

The metrics calculated in the plane η = constant must be converted to the three-dimensional quantities required by the general coordinate transformation, where ξ , η , and ζ are regarded as functions of x , r , and θ . If these metrics are calculated by differencing the coordinates themselves, the easiest way to proceed is to regard the sequence of planes η = constant as determining r , θ , and x for given values of ξ , η , and ζ . The difference formulas then can use:

$$\xi_x = (r_\eta \theta_\zeta - r_\zeta \theta_\eta) / \Delta^{-1}$$

$$\eta_x = (r_\zeta \theta_\xi - r_\xi \theta_\zeta) / \Delta^{-1}$$

$$\zeta_x = (r_\xi \theta_\eta - r_\eta \theta_\xi) / \Delta^{-1}$$

14. Rae, W.J., "A Computer Program for the Ives Transformation in Turbomachinery Cascades", Calspan Report No. 6275-A-3 (November 1980).

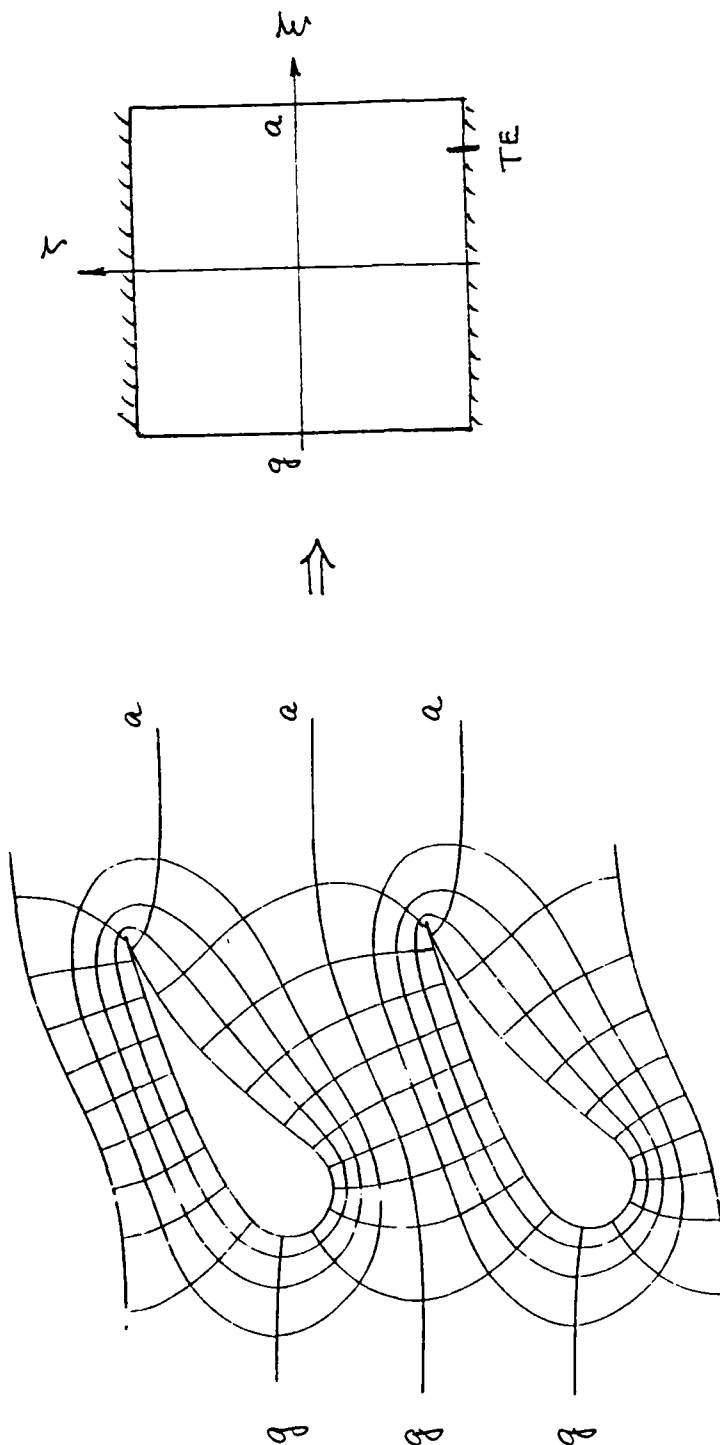


Figure 3. Coordinate Mapping

$$\begin{aligned}
\xi_r &= (\theta_\eta x_\xi - \theta_\xi x_\eta) / \mathcal{D}^{-1} \\
\eta_r &= (\theta_\xi x_\xi - \theta_\xi x_\xi) / \mathcal{D}^{-1} \\
\xi_r &= (\theta_\xi x_\eta - \theta_\eta x_\xi) / \mathcal{D}^{-1} \\
\xi_\theta &= (x_\eta r_\xi - x_\xi r_\eta) / \mathcal{D}^{-1} \\
\eta_\theta &= (x_\xi r_\xi - x_\xi r_\xi) / \mathcal{D}^{-1} \\
\xi_\theta &= (x_\xi r_\eta - x_\eta r_\xi) / \mathcal{D}^{-1}
\end{aligned} \tag{3-19}$$

where

$$\mathcal{D}^{-1} = \frac{\partial(x, r, \theta)}{\partial(\xi, \eta, \zeta)} = \frac{1}{\frac{\partial(\xi, \eta, \zeta)}{\partial(x, r, \theta)}} = \begin{vmatrix} x_\xi & x_\eta & x_\zeta \\ r_\xi & r_\eta & r_\zeta \\ \theta_\xi & \theta_\eta & \theta_\zeta \end{vmatrix} \tag{3-20}$$

is the Jacobian of the transformation.

If the metrics in the $\eta = \text{constant}$ plane are evaluated analytically (for example by conformal mapping, as in the examples used here), then they will contain derivatives of the quantity $r\theta$, taken at constant x and η . In order to extract the desired metrics, the chain rule is used, giving:

$$\begin{aligned}
\left. \frac{\partial \xi}{\partial x} \right)_{r, \theta} &= \left. \frac{\partial \xi}{\partial x} \right)_{y, \eta} + \left. \frac{\partial \xi}{\partial \eta} \right)_{x, y} \left. \frac{\partial \eta}{\partial x} \right)_{r, \theta} \\
\frac{1}{r} \left. \frac{\partial \xi}{\partial \theta} \right)_{x, r} &= \left. \frac{\partial \xi}{\partial y} \right)_{x, \eta} \\
\left. \frac{\partial \xi}{\partial r} \right)_{x, \theta} &= \theta \left. \frac{\partial \xi}{\partial y} \right)_{x, \eta} + \left. \frac{\partial \xi}{\partial \eta} \right)_{x, y} \left. \frac{\partial \eta}{\partial r} \right)_{x, \theta}
\end{aligned} \tag{3-21}$$

and similarly for the derivatives of ζ .

Section 4

BOUNDARY CONDITIONS

The algorithm described in the previous section is implicit, in that the solution vector at all field points is updated by each sequence of three one-dimensional solutions. The values of the solution vector at the grid boundaries can be updated either as part of this implicit scheme, or explicitly. In the former case, special finite-difference versions of the boundary conditions must be developed, and incorporated into each of the one-dimensional solution procedures. In the latter, the updating of the boundary values is separated from the sequence of finite-difference operators used at the field points, and lags one time step behind.

An explicit treatment at the grid boundaries is used in the current program, in order to retain flexibility with regard to the coordinate transformations used. If an implicit treatment of the boundary conditions were used, the specific details of the transformation would have to be built into the main solution algorithm.

Boundary conditions are needed for five variables: three velocity components, and any two thermodynamic variables, for example, pressure, density, rothalpy, total energy). The technique used in external-aerodynamic studies (Reference 8, for example) is to use the surface tangency condition for the three velocities, to extrapolate the density from nearby field points, and to update the pressure using an expression for its normal derivative at the surface.

Surface-Tangency Condition

Since the blade surfaces lie in the planes $\zeta = \text{const.}$, the surface tangency condition is

$$\frac{D\zeta}{Dt} = 0 = \zeta_t + W_x \zeta_x + W_r \zeta_r + \frac{W_\theta}{r} \zeta_\theta = W_3 \quad (4-1)$$

The expressions for the other two contravariant components are then added to this, to give:

$$\begin{pmatrix} \xi_x & \xi_r & \xi_\theta/r \\ \eta_x & \eta_r & \eta_\theta/r \\ \zeta_x & \zeta_r & \zeta_\theta/r \end{pmatrix} \begin{pmatrix} W_x \\ W_r \\ W_\theta \end{pmatrix} = \begin{pmatrix} W_1 - \xi_t \\ W_2 - \eta_t \\ -\zeta_t \end{pmatrix} \quad (4-2)$$

Numerical values for W_1 and W_2 are found at the surface by extrapolation, after which this equation is solved for the surface values of W_x , W_r , and W_θ . A similar procedure is used at the hub and tip, with $W_2 = 0$ in that case.

Normal Pressure - Derivative Relation

The pressure at the surface $\zeta = \text{constant}$ is found by using an expression for $\partial p / \partial \zeta$ to extrapolate from the value a distance $\Delta \zeta$ away from the surface. Two-dimensional counterparts of this derivative expression are given in References 8, 9, 15, with relatively few details about their derivation. The version appropriate to the present problem is:

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15. Pulliam, T.H., and Steger, J.L., "On Implicit Finite-Difference Simulations of Three-Dimensional Flow", AIAA Paper 78-10, (January 1978).

$$\begin{aligned}
& - \frac{\rho}{\mathcal{D}} \left(\zeta_t \tau + W_x \zeta_{x\tau} + W_r \zeta_{r\tau} + W_\theta \left(\frac{\zeta_\theta}{r} \right)_\tau \right) \\
& + \frac{1}{\mathcal{D}} \left\{ \frac{\partial p}{\partial \xi} \left(\zeta_x \xi_x + \zeta_r \xi_r + \frac{\zeta_\theta \xi_\theta}{r^2} \right) + \frac{\partial p}{\partial \eta} \left(\zeta_x \eta_x + \zeta_r \eta_r + \frac{\zeta_\theta \eta_\theta}{r^2} \right) \right. \\
& \quad \left. + \frac{\partial p}{\partial \zeta} \left(\zeta_x^2 + \zeta_r^2 + \frac{\zeta_\theta^2}{r^2} \right) \right. \\
& \quad \left. + \rho W_1 \left(\zeta_x \frac{\partial W_x}{\partial \xi} + \zeta_r \frac{\partial W_r}{\partial \xi} + \frac{\zeta_\theta}{r} \frac{\partial W_\theta}{\partial \xi} \right) + \rho W_2 \left(\zeta_x \frac{\partial W_x}{\partial \eta} + \zeta_r \frac{\partial W_r}{\partial \eta} + \frac{\zeta_\theta}{r} \frac{\partial W_\theta}{\partial \eta} \right) \right\} \\
& = \left\{ \rho \zeta_r [W_\theta + \omega r]^2 - \frac{\zeta_\theta}{r} \rho W_r [W_\theta + 2\omega r] \right\} / r \mathcal{D} \tag{4-3}
\end{aligned}$$

This is to be regarded as an expression for $\partial p / \partial \zeta$ on the blade surfaces $\zeta = \text{constant}$, with all other quantities either known or found by extrapolation. Its usefulness lies in the fact that it contains no time derivatives of the dependent variables. The derivation of this equation is achieved by summing the three components of the momentum equation, multiplied respectively by ζ_x , ζ_r and ζ_θ / r . The resulting equation is then arranged into four groups, containing derivatives with respect to τ , ξ , η and ζ respectively. Within each of these groups, appropriate terms are added and subtracted so as to form the quantity W_3 , which is zero. Expansion of certain of the ξ , η and ζ derivatives then leads to cancellation of a number of terms involving the product of the pressure times derivatives of the metrics. Other terms of this type, which do not cancel, can be rearranged by noting the property of the Jacobian that

$$\frac{1}{\mathcal{D}} \frac{\partial \mathcal{D}}{\partial \alpha} = \frac{\partial}{\partial \xi} \frac{\partial \xi}{\partial \alpha} + \frac{\partial}{\partial \eta} \frac{\partial \eta}{\partial \alpha} + \frac{\partial}{\partial \zeta} \frac{\partial \zeta}{\partial \alpha} \tag{4-4}$$

where α is x , r or θ . After these simplifications have been made, four of the remaining terms can be recognized as the continuity equation; removing these terms takes out the only remaining terms involving τ -derivatives of

the dependent variables. Finally, it is necessary to add and subtract certain derivatives of $1/r$ and to use the relation

$$\xi_{\theta} \frac{\partial}{\partial \xi} \left(\frac{1}{r} \right) + \eta_{\theta} \frac{\partial}{\partial \eta} \left(\frac{1}{r} \right) + \xi_{\theta} \frac{\partial}{\partial \xi} \left(\frac{1}{r} \right) = \frac{\partial}{\partial \theta} \left(\frac{1}{r} \right) = 0 \quad (4-5)$$

A similar relation for the pressure gradient normal to the hub and shroud can be found, by summing the momentum equations, multiplied respectively by η_x , η_r , and η_{θ} (and then using $\eta_{\theta} = 0$). The result is

$$\begin{aligned} & - \frac{\rho}{\mathcal{D}} (\eta_{t\tau} + w_x \eta_{x\tau} + w_r \eta_{r\tau}) \\ & + \frac{1}{\mathcal{D}} \left\{ \frac{\partial p}{\partial \xi} (\eta_x \xi_x + \eta_r \xi_r) + \frac{\partial p}{\partial \eta} (\eta_x^2 + \eta_r^2) + \frac{\partial p}{\partial \xi} (\eta_x \xi_x + \eta_r \xi_r) \right. \\ & \left. + \rho w_1 (\eta_x \frac{\partial w_x}{\partial \xi} + \eta_r \frac{\partial w_r}{\partial \xi}) + \rho w_3 (\eta_x \frac{\partial w_x}{\partial \xi} + \eta_r \frac{\partial w_r}{\partial \xi}) \right\} \\ & = \rho \eta_r [w_{\theta} + w_r]^2 / r \mathcal{D} \end{aligned} \quad (4-6)$$

At $\xi = \pm 1$, a symmetry condition is imposed, explicitly:

$$U(\pm 1, \eta, \xi) = U(\pm 1, \eta - \xi) \quad (4-7)$$

Kutta Condition

The trailing-edge region is treated in the present work as though the trailing edge were a cusp, i.e., the pressures and flow angles leaving either side of the trailing edge are required to be the same, although the velocity magnitudes may be unequal. For small trailing-edge included angles,

the application of the flow tangency condition on the blades would be expected to produce nearly equal flow angles; thus in the present work, the pressures were matched, by updating the quantity K at the two points denoted by + and - in Fig. 4. Thus:

$$p = (\gamma - 1) \left[K - \frac{U_1}{2} (W^2 - (\omega r)^2) \right]$$

$$p_+ - p_- = (\gamma - 1) \left[K^+ - K^- - \frac{1}{2} (U_1^+ W_+^2 - U_1^- W_-^2) \right] = 0 \quad (4-8)$$

In order to set K^+ and K^- , define

$$\bar{K} \equiv \frac{1}{2} (K^+ + K^-)_{ext} \quad (4-9)$$

where the notation $(\)_{ext}$ signifies that these are values extrapolated to the surface. Then calculate:

$$(\Delta K)_{new} \equiv (K^+ - K^-)_{new} = \frac{1}{2} [U_1^+ W_+^2 - U_1^- W_-^2] \quad (4-10)$$

$$K_{new}^{\pm} = \bar{K} \pm \frac{1}{2} (\Delta K)_{new}$$

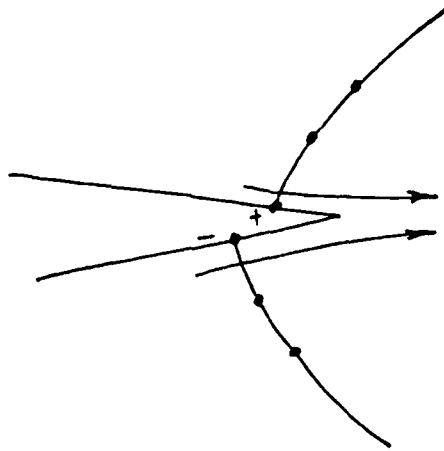


Figure 4. Trailing-Edge Notation

Conditions at Infinity

The points at infinity ($\xi = 0, \xi = \pm 1$) are excluded from the grid by using an even number of grid points in the ξ - direction:

$$\xi = (L-1) \Delta \xi, \quad L = 1, 2, \dots, \text{LMX}; \quad \text{LMX even, } \Delta \xi = 2/\text{LMX}-1 \quad (4-11)$$

The points at $\xi = \pm 1, L = L^+$ and $L = L^-$ (where L^- is the integer part of $(\text{LMX} + 1)/2$, and $L^+ = L^- + 1$ - see Fig. 3 -) are assigned fixed values at the beginning of the calculation, and are not changed thereafter. In particular, care is taken to avoid differencing across these points when applying the symmetry condition at $\xi = \pm 1$.

The question of how to select the values that are assigned at plus and minus infinity is a serious problem in itself. The literature contains a

number of papers which use time-marching methods to solve the Euler equations.¹⁶⁻²⁰ In several of these, the method of characteristics is applied to the equations (in nonconservative form) in order to calculate the solution at the grid boundaries. In other papers, certain dependent variables such as the pressure or outlet flow angle are prescribed, and the remaining variables deduced from these. However, there does not exist at present a complete treatment of this "Trefftz-plane" problem, connecting the far-field solution to conditions at the blades, and giving the relations between the dependent variables themselves.²¹ A prominent example of what is missing from the current literature is the connection between the pressure far downstream and the trailing-edge conditions: in the nonlinear small-disturbance theory, the imposition of the Kutta condition at the trailing edge uniquely determines the circulation at that spanwise station, which in turn defines the pressure rise and turning angle that must be reached far downstream of the blade row.^{1,4} The extension of this relationship to the case of the full Euler equations has not been made. Thus, for example, it must be presumed

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16. McDonald, P.W., "The Computation of Transonic Flow Through Two-Dimensional Gas Turbine Cascades", American Society of Mechanical Engineers, Paper 71-GT-89, 1971.
 17. Gopalakrishnan, S. and Bozzola, R., "Computation of Shocked Flows in Compressor Cascades", American Society of Mechanical Engineers, Paper 72-GT-31, 1972.
 18. Kurzrock, J.W. and Novick, A.S., "Transonic Flow Around Rotor Blade Elements", Transactions of the American Society of Mechanical Engineers, Vol. 97, December 1975, pp. 598-607.
 19. Thompkins, William T., Jr., "An Experimental and Computational Study of the Flow in a Transonic Compressor Rotor", MIT Gas Turbine Laboratory Report No. 129 (May 1976).
 20. Veuillot, J.P., "Calculation of the Quasi-Three Dimensional Flow in a Turbomachine Blade Row", Trans. ASME (A), Journal of Engineering for Power 99, (1977) 53-62.
 21. Karamcheti, K., Principles of Ideal-Fluid Aerodynamics, Wiley and Sons, New York (1966) Section 19.4.

that the assignment of an outlet flow angle implies a violation of the Kutta condition, and this in turn renders the circulation and pressure rise non-unique. It is not clear from the published literature how these problems are resolved in current computer codes.

The present research has not addressed these questions, because of the instabilities encountered in the process of developing the program. Instead, a set of boundary values, described below, was assigned at downstream infinity. These values were adequate for use in the early development of the computer program, but will have to be replaced by a more exact formulation, after the grid-related oscillations have been removed.

The specific choices for the variables at downstream infinity were made as follows: the static pressure ratio across the blade row was assigned as an input, and the density was found from the isentropic relation. The area ratio between outlet and inlet was assigned, and from this the axial velocity component w_x was found by conserving the mass flow $\rho A w_x$. The radial velocity w_r was set equal to zero, and the circumferential component w_θ was chosen, following Reference 19, as that value which gives a uniform static pressure, i.e., radial equilibrium requires

$$\frac{1}{\rho} \frac{\partial p}{\partial r} = 0 = \frac{1}{r} [w_\theta - \omega r]^2, \quad \text{or } w_\theta = \omega r \quad (4-12)$$

The value of the quantity K then follows from these specifications.

Wake Conditions

Whenever a compressor blade is acted on by a lift that varies with radius, a sheet of vorticity will be shed from the trailing edge. The origins of this vortex sheet can be seen in the nonlinear small-disturbance results of Reference 4; Figure 11 of that paper shows the distributions of radial velocity at a sixty-percent chord location. These distributions retain the same qualitative behavior all the way to the trailing edge, i.e., they reveal a discontinuity at the trailing edge, which trails downstream as a vortex sheet.

In the small-disturbance potential-theory approximation, the trailing vortex sheets are assumed to lie on the helical surfaces defined by the inlet flow. In a full nonlinear treatment, they must be allowed to deform, away from these surfaces, as they move downstream. This problem has been studied recently in a series of papers by McCune and Hawthorne.²²⁻²⁴ These papers constitute a basis on which to model the vortex-sheet trajectories in a finite-difference code. For example, the discontinuities in radial velocity that occur at the trailing edge would have to be inserted at the trailing-edge location, and the locus of this discontinuity would have to be followed downstream. The conformal transformation used in the present work is not well suited for doing so, however, since the path followed by the trailing vortex sheet in the ξ, ζ plane (Figure 3) is a line that leaves the trailing-edge image, and spirals around the image of the point at downstream infinity. It would be virtually impossible to use a grid that is fine enough to resolve these discontinuities numerically. In order to facilitate the resolution of the vortex-sheet behavior, it would be necessary to use a different coordinate transformation, in which the path of the vortex sheet is not as convoluted as it is in the case of the Ives mapping.

As an alternative, it might be possible to trace the magnitude and location of these discontinuities by "floating vortex-sheet fitting", in analogy to the procedure of floating shock fitting. The development of such a procedure was not considered in the present research.

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Section 5

RESULTS

A two-dimensional cascade was chosen for the purpose of debugging the program. The blade geometry, shown in Figure 5, is the same as used by Rae and Homicz (Reference 25). It was not chosen on the basis of any design method, but only for the purpose of facilitating a demonstration calculation. The solidity is moderate, and the large leading-edge radius minimizes strong flow field gradients in that region.

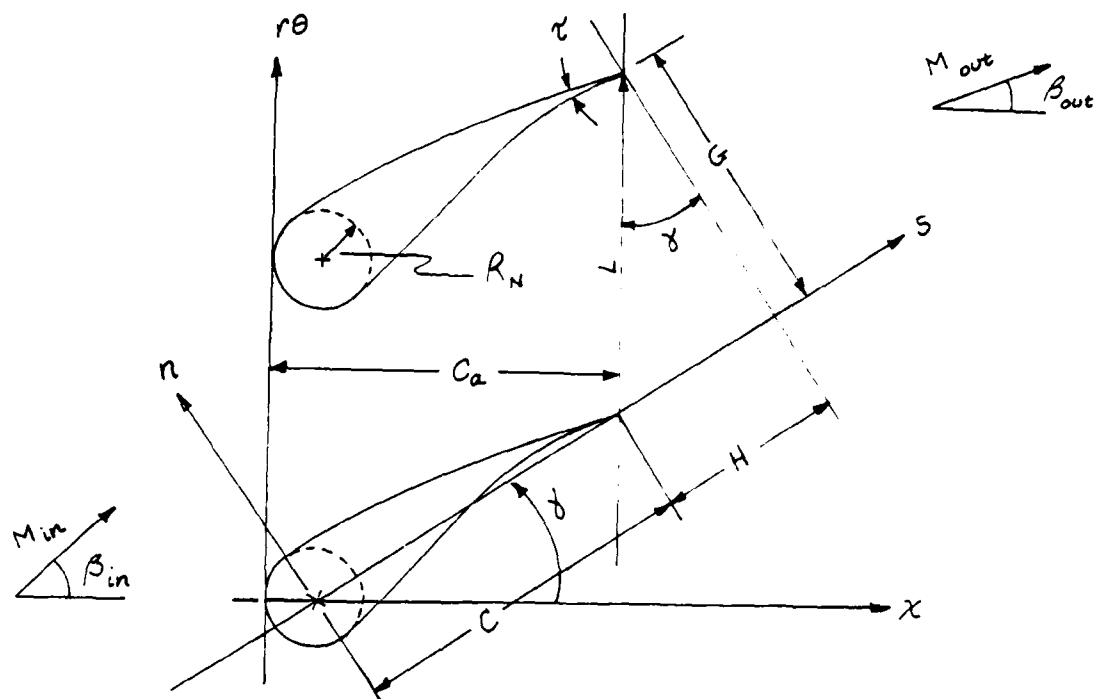
In order to do a two-dimensional case, the hub and shroud radii were taken to be $r_h/c_a = 99.5$, $r_t/c_a = 100.5$, and the calculations were done at $r/c_a = 100$. The inlet relative Mach number and flow angle were taken as 0.5 and 33° . All quantities were made dimensionless by dividing by the appropriate combination of the density and axial velocity component far upstream, ρ_∞ and U_∞ , and the axial projection of the chord, C_a . Thus, for example, the dimensionless angular velocity was input as

$$\frac{\omega C_a}{U_\infty} = \frac{\omega r}{a_\infty} \cdot \frac{C_a}{r} \cdot \frac{a_\infty}{U_\infty} = \sin \beta_{in} \cdot \frac{1}{100} \cdot \frac{1}{\cos \beta_{in}} = \frac{\tan(33^\circ)}{100} \quad (5-1)$$

The specific-heat ratio was taken as 1.4, and the grid sizes in the transformed plane as $KMX = 11$, $LMX = 10$, giving $\Delta\xi = 0.2$, $\Delta\zeta = 2/9$.

To start the calculations, an input tape was prepared, containing values of the metrics, the Jacobian, and the radii at each grid point. On the first run, all dependent variables were initialized at their upstream values. At the end of each run, the metrics, Jacobians, and radii were

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25. Rae, W.J., and Homicz, G.F., "A Rectangular-Coordinate Method For Calculating Nonlinear Transonic Potential Flowfields in Compressor Cascades", AIAA Paper 78-248 (January 1978).



$$\frac{C_a}{L} = 1, \quad \delta = 32.91^\circ, \quad R_N/C_a = 0.15, \quad \tau = 13.3^\circ$$

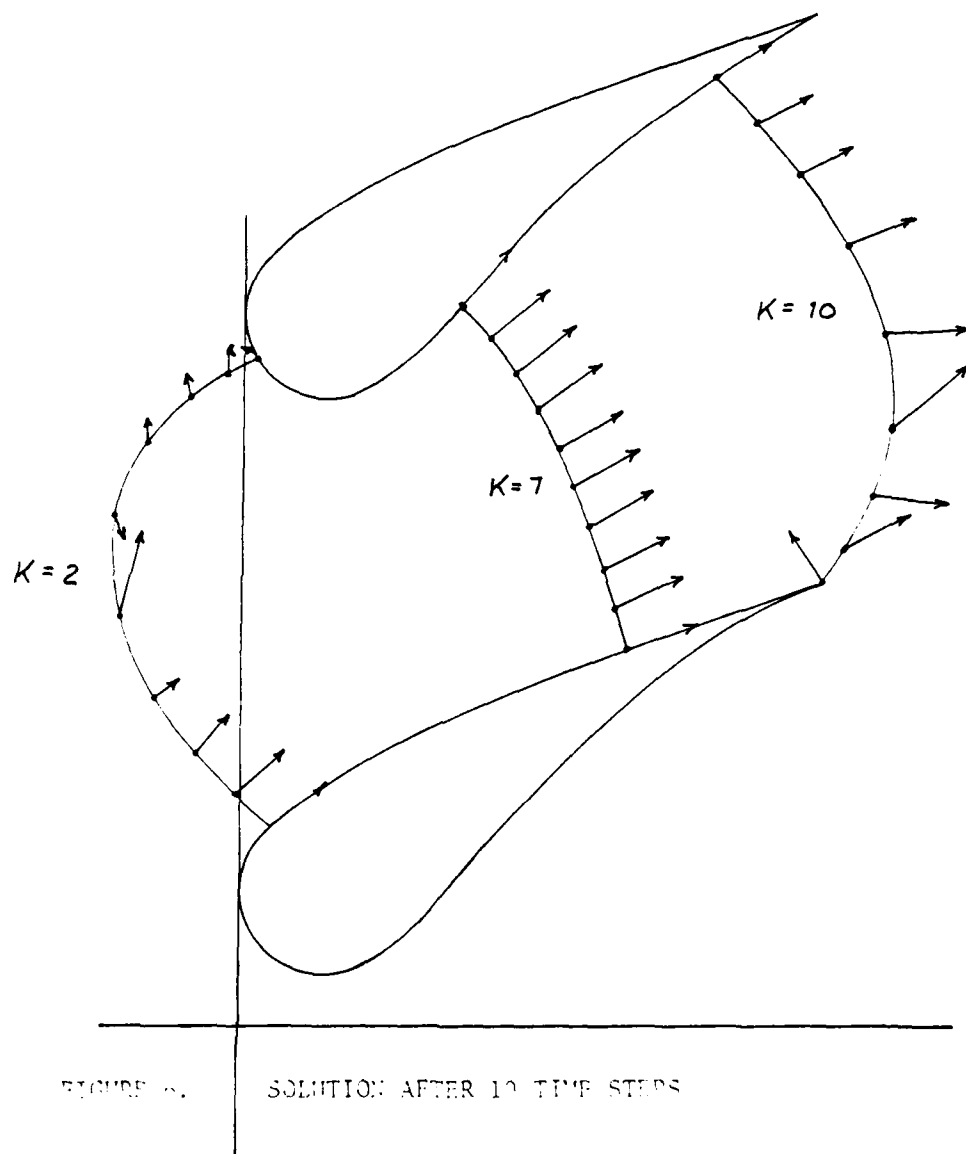
FIGURE 5. CASCADE GEOMETRY

rewritten on a new tape, along with values of the solution at the last time step. This tape could then be used to start the next series of time steps. The maximum time step allowed by the CFL condition (see Section 3) was calculated at the end of each step; all of the results discussed below were calculated with a time step equal to half this value, and with damping coefficients ϵ_ρ and ϵ_θ equal to the (dimensionless) value of the time step. The static pressure ratio was assigned as 1.1, and the area ratio as 1.0.

Figure 6 shows results for the velocity field after ten time steps. In the guided channel between the blades, the results conform to what would be expected, but at the stations $K = 2$ and $K = 10$, there are very large oscillations. On this coarse grid, these two stations are the outermost ones at which implicit calculations are done; the stations $K = 1$ and 11 form the boundaries of the computational grid, and their values are updated explicitly one time step behind.

The oscillations at the station $K = 2$ are due to the metric singularities at $K = 1$; the image of the point at upstream infinity is at $K = 1$, and midway between the central pair of L - values (see Figure 3). Even though the metrics at $K = 1$, $L = L^+$ and L^- are finite, nevertheless their gradients are so steep at those points that they destabilize the solution. This effect can be displayed clearly by the results of a calculation in which the solution is initialized to the freestream values, and then advanced by a single time step. For the two-dimensional case, the basic equation (1-1) becomes

$$U_\infty \frac{\partial}{\partial \tau} \left(\frac{1}{\mathcal{D}} \right) = 0 = - \left\{ E_\infty \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_x}{\mathcal{D}} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\zeta_x}{\mathcal{D}} \right) \right] + \frac{G_\infty}{r} \left[\frac{\partial}{\partial \xi} \left(\frac{\xi_\theta}{\mathcal{D}} \right) + \frac{\partial}{\partial \zeta} \left(\frac{\zeta_\theta}{\mathcal{D}} \right) \right] \right\}$$

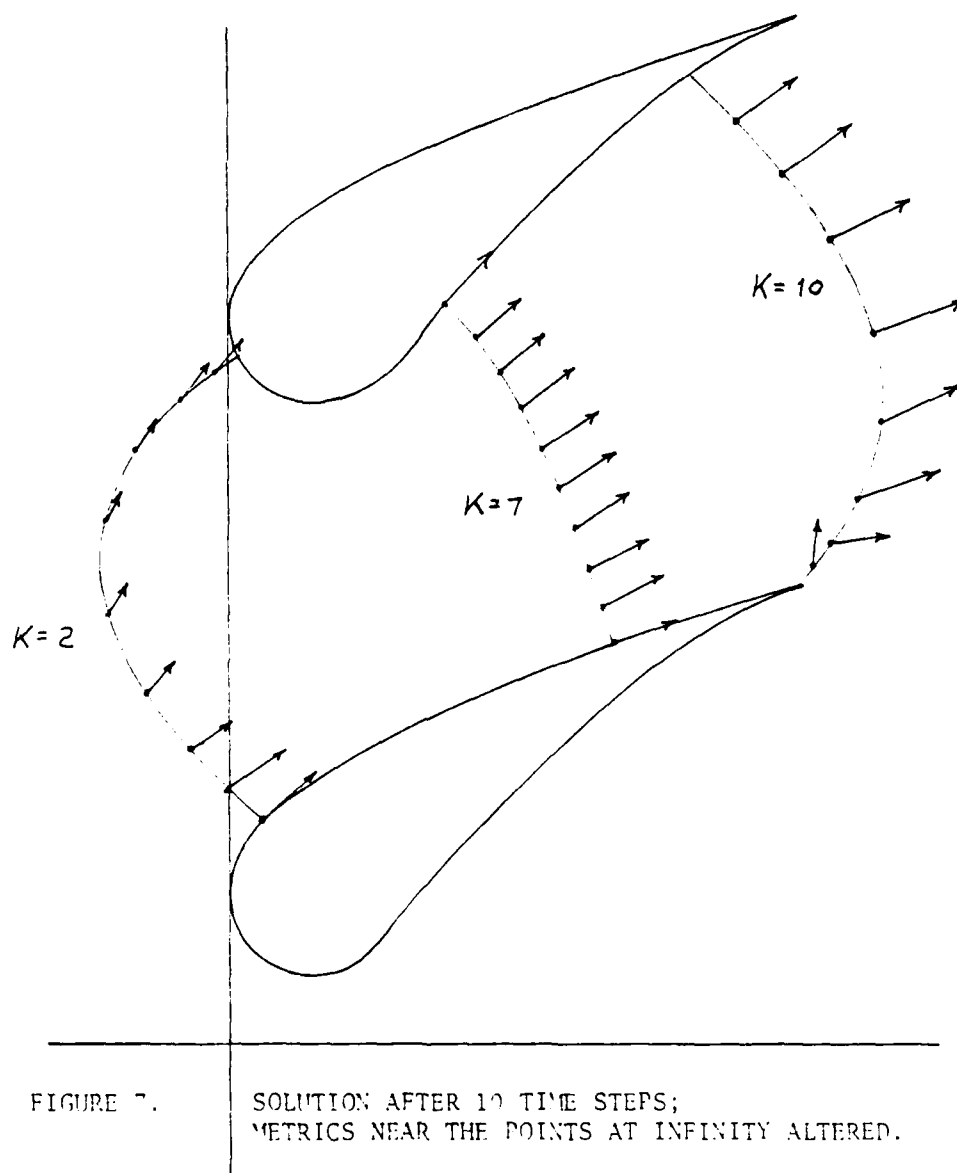


But the quantities in the square brackets on the right side of this equation are each zero (as can be verified from Eq. 3-20, with $\partial r / \partial \eta = 1$). This result is exact, analytically, but when evaluated numerically, especially on a coarse grid, the result is nonzero, and of a magnitude consistent with the magnitude of the oscillations that develop. Part of the problem, in the present case, is due to the use of analytic formulas for the metrics; it is pointed out in Reference 8 that metrics which are generated numerically, by differencing the coordinate mapping itself, are actually less sensitive to this problem than the analytic metrics.

In an attempt to alleviate this problem, the values of the metrics at $K = 1$, $L = L^-$ and L^+ were changed, as follows: the value of $\xi_\theta / r)_{K=1}^{L=L^-}$ which is equal to $-\xi_\theta / r)_{K=1}^{L=L^+}$, was chosen so as to make the factor multiplying G_∞ numerically equal to zero. Next, the value of $\xi_x)_{K=1}^{L=L^-}$, which is equal to $-\xi_x)_{K=1}^{L=L^+}$, was changed so that the factor multiplying E_∞ would have the same value at $K = 1$, $L = L^-$ and L^+ . The result of this modification is shown in Figure 7, at the tenth time step. Comparison with Figure 6 shows that a considerable smoothing of the flow pattern at $K = 1$ was achieved.

Analogous modifications were made at $K = KMX$, but these did not yield the same degree of success, presumably because the solution on this line is also affected by the singular region near the trailing edge. Several attempts were made to overcome this problem, by using extrapolation to update the points near the trailing edge. These attempts were not successful. Moreover, this approach is difficult to justify, since points on the surface near the trailing edge are very important to the solution, in that they are the ones used in applying the Kutta condition, as well as in enforcing the surface tangency condition. Any extrapolation procedure alters the role of these boundary points, making them dependent on the field behavior, rather than the other way around.

The metric-singularity problems encountered in this research are aggravated by the use of the coarse grid, and by the fact that the grid wraps



around the trailing edge of the blades. While the use of a finer grid might alleviate the situation somewhat, it was not used, since it appeared that lengthier calculations would not be justified until the more basic cause - the trailing-edge singularity - was eliminated.

Section 6

CONCLUDING REMARKS

The experience gained in this study suggests strongly that type of grid used (where the metrics are evaluated from analytic formulas, and the grid is wrapped around the trailing edge) is not suitable for use with the implicit time-marching algorithm. The evidence is not conclusive, however; it may be that use of a finer grid and some other special treatment of the metrics in the region of the singularities could stabilize the calculations. However, a preferable course, for future developments, appears to be the use of grids which are free from singularities, especially in the trailing-edge region.

After the grid-induced instabilities have been removed, a number of other problems will remain. These problems were not considered in depth during this research, because of the amount of effort devoted to the instability problems. Among the topics that will have to be considered are the Kutta and far-field conditions, the location and strength of the trailing vortex sheets, and the interaction between all of these. Also remaining is the problem of shock capturing in genuinely transonic flows, which may require alterations in the difference formulas used. Finally, there are several advances in the time-marching algorithm that have taken place during the period of this research, such as the technique of flux vector splitting.²⁶ These should be considered for incorporation in the numerical method.

26. Steger, J.L., and Warming, R.F., "Flux Vector Splitting of the Inviscid Gasdynamic Equations with Application to Finite Difference Methods", NASA TM 78605 (July 1979).

APPENDIX

The dependent variables are defined as:

$$U = \begin{bmatrix} p \\ \rho w_x \\ \rho w_r \\ \rho w_\theta \\ K \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} \quad E(U) = \begin{bmatrix} \rho w_x \\ p + \rho w_x^2 \\ \rho w_x w_r \\ \rho w_x w_\theta \\ w_x (K + p) \end{bmatrix} = \begin{bmatrix} u_2 \\ p + u_2^2/u_1 \\ u_2 u_3/u_1 \\ u_2 u_4/u_1 \\ \frac{u_2}{u_1} (u_5 + p) \end{bmatrix}$$

$$F(U) = \begin{bmatrix} \rho w_r \\ \rho w_r w_x \\ p + \rho w_r^2 \\ \rho w_r w_\theta \\ w_r (K + p) \end{bmatrix} = \begin{bmatrix} u_3 \\ u_2 u_3/u_1 \\ p + u_3^2/u_1 \\ u_3 u_4/u_1 \\ \frac{u_3}{u_1} (u_5 + p) \end{bmatrix} \quad G(U) = \begin{bmatrix} \rho w_\theta \\ \rho w_\theta w_x \\ \rho w_\theta w_r \\ p + \rho w_\theta^2 \\ w_\theta (K + p) \end{bmatrix} = \begin{bmatrix} u_4 \\ u_2 u_4/u_1 \\ u_3 u_4/u_1 \\ p + u_4^2/u_1 \\ \frac{u_4}{u_1} (u_5 + p) \end{bmatrix}$$

$$H(U) = \begin{bmatrix} 0 \\ 0 \\ p + \rho (w_\theta + w_r)^2 \\ -\rho w_r (2w_r + w_\theta) \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p + u_1 (u_4/u_1 + w_r)^2 \\ -u_3 (2w_r + u_4/u_1) \\ 0 \end{bmatrix}$$

where $p = (\gamma - 1) \left[u_5 - \frac{u_2^2 + u_3^2 + u_4^2}{2u_1} + \frac{(w_r)^2}{2} u_1 \right]$.

Thus the matrices A, B, C, and D are

$$A_{ij} = \frac{\partial E_i}{\partial u_j}, \quad B_{ij} = \frac{\partial F_i}{\partial u_j}, \quad C_{ij} = \frac{\partial G_i}{\partial u_j}, \quad D_{ij} = \frac{\partial H_i}{\partial u_j}.$$

APPENDIX (continued)

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ A_{21} & (3-\delta)\frac{u_2}{u_1} & -(\delta-1)\frac{u_3}{u_1} & -(\delta-1)\frac{u_4}{u_1} & \delta-1 \\ -\frac{u_2 u_3}{u_1^2} & \frac{u_3}{u_1} & \frac{u_2}{u_1} & 0 & 0 \\ -\frac{u_2 u_4}{u_1^2} & \frac{u_4}{u_1} & 0 & \frac{u_2}{u_1} & 0 \\ A_{51} & A_{52} & -(\delta-1)\frac{u_2 u_3}{u_1 u_1} & -(\delta-1)\frac{u_2 u_4}{u_1 u_1} & \delta\frac{u_2}{u_1} \end{bmatrix}$$

where

$$A_{21} = (\delta-1) \left\{ \frac{u_2^2 + u_3^2 + u_4^2}{2u_1^2} + \frac{(\omega r)^2}{2} \right\} - \frac{u_2^2}{u_1^2}$$

$$A_{51} = -\frac{u_2(u_3 + p)}{u_1^2} + (\delta-1) \frac{u_2}{u_1} \left\{ \frac{u_2^2 + u_3^2 + u_4^2}{2u_1^2} + \frac{(\omega r)^2}{2} \right\}$$

$$A_{52} = \frac{u_3 + p}{u_1} - (\delta-1) \left(\frac{u_2}{u_1} \right)^2$$

$$B = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ -\frac{u_2 u_3}{u_1^2} & \frac{u_3}{u_1} & \frac{u_2}{u_1} & 0 & 0 \\ B_{31} & -(\delta-1)\frac{u_2}{u_1} & (3-\delta)\frac{u_3}{u_1} & -(\delta-1)\frac{u_4}{u_1} & \delta-1 \\ -\frac{u_3 u_4}{u_1^2} & 0 & \frac{u_4}{u_1} & \frac{u_3}{u_1} & 0 \\ B_{51} & -(\delta-1)\frac{u_3 u_2}{u_1 u_1} & B_{53} & -(\delta-1)\frac{u_3 u_4}{u_1 u_1} & \delta\frac{u_3}{u_1} \end{bmatrix}$$

where

APPENDIX (continued)

$$B_{31} = (\gamma-1) \left\{ \frac{u_2^2 + u_3^2 + u_4^2}{2u_1^2} + \frac{(\omega r)^2}{2} \right\} - \frac{u_3^2}{u_1^2}$$

$$B_{51} = -\frac{u_3(u_5 + p)}{u_1^2} + (\gamma-1) \frac{u_3}{u_1} \left\{ \frac{u_2^2 + u_3^2 + u_4^2}{2u_1^2} + \frac{(\omega r)^2}{2} \right\}$$

$$B_{53} = \frac{u_5 + p}{u_1} - (\gamma-1) \left(\frac{u_3}{u_1} \right)^2$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ \frac{-u_2 u_4}{u_1^2} & \frac{u_4}{u_1} & 0 & \frac{u_2}{u_1} & 0 \\ \frac{-u_3 u_4}{u_1^2} & 0 & \frac{u_4}{u_1} & \frac{u_3}{u_1} & 0 \\ C_{41} & -(\gamma-1) \frac{u_2}{u_1} & -(\gamma-1) \frac{u_3}{u_1} & (3-\gamma) \frac{u_4}{u_1} & \gamma-1 \\ C_{51} & -(\gamma-1) \frac{u_4 u_2}{u_1 u_1} & -(\gamma-1) \frac{u_4 u_3}{u_1 u_1} & C_{54} & \gamma \frac{u_4}{u_1} \end{bmatrix}$$

where

$$C_{41} = (\gamma-1) \left\{ \frac{u_2^2 + u_3^2 + u_4^2}{2u_1^2} + \frac{(\omega r)^2}{2} \right\} - \frac{u_4^2}{u_1^2}$$

$$C_{51} = -\frac{u_4(u_5 + p)}{u_1^2} + (\gamma-1) \frac{u_4}{u_1} \left\{ \frac{u_2^2 + u_3^2 + u_4^2}{2u_1^2} + \frac{(\omega r)^2}{2} \right\}$$

$$C_{54} = \frac{u_5 + p}{u_1} - (\gamma-1) \left(\frac{u_4}{u_1} \right)^2$$

APPENDIX (continued)

$$D = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ D_{31} & -(\gamma-1)\frac{u_2}{u_1} & -(\gamma-1)\frac{u_3}{u_1} & D_{34} & \gamma-1 \\ \frac{u_3 u_4}{u_1^2} & 0 & -(2\omega r + \frac{u_4}{u_1}) & -\frac{u_3}{u_1} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where

$$D_{31} = (\gamma-1) \left\{ \frac{u_2^2 + u_3^2 + u_4^2}{2u_1^2} + \frac{(\omega r)^2}{2} \right\} + (\omega r)^2 - \left(\frac{u_4}{u_1} \right)^2$$

$$D_{34} = (\gamma-1) \frac{u_4}{u_1} + 2\omega r$$

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